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An algebraic description of the elliptic cohomology of classifying spaces

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Abstract

Let G be a finite group of order $|G|$ odd and let $\mathscr{E}\ell\ell^*(-)\otimes\mathbb{Z}[1/|G|]$ denote elliptic cohomology tensored by $\mathbb{Z}[1/|G|]$. Then we give a description of $\mathscr{E}\ell\ell^*(E(N,G)\times_N X)\otimes\mathbb{Z}[1/|G|]$, where N is a normal subgroup of G, $E(N,G)$ is the universal N-free G space and X is any finite G-CW complex where N acts freely. We explain how some of the results of Hopkins–Kuhn–Ravenel can be recovered for our results. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [12] we defined, for any finite group G of odd order $|G|$, a multiplicative G-equivariant cohomology theory $\mathscr{E}\ell\ell_G^*$. This theory is a G-equivariant generalization of the cohomology theory $X\to\mathscr{E}\ell\ell^*(X)\otimes_{\mathbb{Z}}\mathbb{Z}[1/|G|]$, where $\mathscr{E}\ell\ell^*$ is the *elliptic cohomology* of Landweber, Ravenel and Stong [21]. For this reason we called $\mathscr{E}\ell\ell_G^*$ *equivariant elliptic cohomology*. If X is a finite G-CW complex, then $\mathscr{E}\ell\ell_G^*(X)$ is defined by the equality

$$\mathscr{E}\ell\ell_G^*(X)=MSO_G^*(X)\bigotimes_{MSO_G^*}\mathscr{E}\ell\ell_G^*, \tag{1.1}$$

where $MSO_G^*(-)$ is the integer graded version of the homotopy theoretic-oriented equivariant cobordism functor of [10], $MSO_G^*=MSO_G^*(pt)$, and $\mathscr{E}\ell\ell_G^*$ is a graded ring closely related to the moduli space of G-coverings, in the sense of algebraic geometry,

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of Jacobi quartics. The ring $\mathscr{E}\ell\ell_G^*$ in (1.1) is considered as an algebra over MSO_G^* via a ring homomorphism $MSO_G^* \xrightarrow{\Phi_G} \mathscr{E}\ell\ell_G^*$. We called Φ_G the *twisted elliptic genus*.

In this paper we shall give a description of $\mathscr{E}\ell\ell_G^*(E\mathscr{F} \times X)$, where X is a finite G-CW complex, $\mathscr{F}$ is any family of subgroups of G, and $E\mathscr{F}$ is the universal $\mathscr{F}$-free G-space. Applying this description to suitable families of subgroups we shall obtain in particular a description of the elliptic cohomology (tensored by $\mathbb{Z}[1/|G|]$) of the classifying spaces BG and $B(N,G)$, where N is any normal subgroup of G.

The layout of this paper is as follows. In Sections 2 and 3 we shall describe briefly the results of [12]. Firstly, we shall discuss different aspects of the coefficient ring $\mathscr{E}\ell\ell_G^*$. Secondly, we shall study the cohomological properties of the functor $X \to \mathscr{E}\ell\ell_G^*(X)$. The most important section of the paper is Section 4, where we shall prove the main results of the paper. Finally, we shall explain how some of the rings of generalized characters of Hopkins, Kuhn and Ravenel arise naturally from our results.

2. The ring $\mathscr{E}\ell\ell_G^*$ and its ideals

2.1. Basic definitions

We shall denote the complex upper half plane by $\mathfrak{h}_+$, and we shall write $\Gamma_0(2)$ for the group

$$\left\{ \begin{pmatrix} a & b \\ e & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, e \equiv 0 \,(\mathrm{mod}\, 2) \right\}. \tag{2.1}$$

If G is a finite group of odd order and we write TG for the set

$$\{(g_1,g_2) \in G \times G \mid g_1 g_2 = g_2 g_1\}, \tag{2.2}$$

then the group $\Gamma_0(2) \times G$ acts on the left on $TG \times \mathfrak{h}_+$ by

$$\left(\begin{pmatrix} a & b \\ e & d \end{pmatrix}, g \right) \times ((g_1,g_2),\tau) \xrightarrow{\rho} \left(g(g_1^d g_2^{-e}, g_1^{-b} g_2^a) g^{-1}, \frac{a\tau+b}{e\tau+d} \right). \tag{2.3}$$

The action ρ induces, for each $k \in \mathbb{Z}$, an action ρ_k of $\Gamma_0(2) \times G$ on the ring of functions $\vartheta : TG \times \mathfrak{h}_+ \to \mathbb{C}$. The action ρ_k is defined by

$$\rho_k \left(\begin{pmatrix} a & b \\ e & d \end{pmatrix}, g \right) \vartheta((g_1,g_2),\tau) = (e\tau+d)^{-k} \vartheta \left(g(g_1^d g_2^{-e}, g_1^{-b} g_2^a) g^{-1}, \frac{a\tau+b}{e\tau+d} \right). \tag{2.4}$$

We shall say that a function $\vartheta : TG \times \mathfrak{h}_+ \to \mathbb{C}$ is *holomorphic* if and only if for each element $(g_1,g_2) \in TG$ the function $\vartheta((g_1,g_2),-) : \mathfrak{h}_+ \to \mathbb{C}$ is holomorphic in the usual sense of the word. It is easy to see that the action ρ_k preserves the holomorphic functions.

Definition 2.1. The group $\mathscr{E}\ell\ell_G^{-2k}$ is the Abelian group whose elements are the holomorphic functions $\vartheta : TG \times \mathfrak{h}_+ \to \mathbb{C}$ that satisfy the following conditions:

(1) $\rho_k\left(\left(\begin{smallmatrix} a & b \\ e & d \end{smallmatrix}\right), g\right)\ \vartheta = \vartheta, \quad \forall\left(\left(\begin{smallmatrix} a & b \\ e & d \end{smallmatrix}\right), g\right) \in \Gamma_0(2) \times G$;

(2) for each $(g_1, g_2) \in TG$ the functions

$$\vartheta((g_1,g_2),-) : \mathfrak{h}_+ \to \mathbb{C}, \text{ and } \vartheta'((g_1,g_2),\tau) = \tau^{-k}\vartheta((g_1,g_2),-1/\tau)$$

have power series expansions at $i\infty$ of the form

$$\vartheta((g_1,g_2),\tau) = \sum_{n\geq K} a_n q^{n/|g_1|}, \qquad \vartheta'((g_1,g_2),\tau) = \sum_{n\geq K} b_n q^{n/|g_1|},$$

where $K \in \mathbb{Z}$, $q = \exp\{2\pi i\tau\}$, and $a_n, b_n \in \mathbb{Z}\left[\frac{1}{2}, 1/|G|, \exp\{2\pi i/|g_1 g_2|\}\right]$;

(3) Let $C_{g_1}(G)$ be the centralizer of g_1 in G, and let $\psi = \exp\{2\pi i/|C_{g_1}(G)|\}$. If n and $|C_{g_1}(G)|$ are coprime, and σ_n is the ring automorphism of $\mathbb{Z}[1/|G|, \psi]$ defined by $\sigma_n(\psi) = \psi^n$, then

$$\sigma_n(a_m(g_1,g_2)) = a_m(g_1,g_2^n), \quad \sigma_n(b_m(g_1,g_2)) = b_m(g_1,g_2^n). \tag{2.5}$$

The group structure in $\mathscr{E}\ell\ell_G^*$ is induced by the sum of functions.

Remark 2.2. The second condition in the definition of $\mathscr{E}\ell\ell_G^{-2k}$ is, from the point of view of modular forms, the strongest possible integrality condition [18, p. 80, (Ka-12)].

Remark 2.3. The action of $(\mathbb{Z}/|C_{g_1}(G)|\mathbb{Z})^*$ on the group $C_{g_1}(G)$ appears in representation theory [16]. The action of σ_n on the coefficients a_m is associated to the usual Galois action of the group $(\mathbb{Z}/|C_{g_1}(G)|)^*$ on modular forms of higher level [22, Ch. 6, Section 3].

The third condition in Definition 2.1 implies that for all the elements $g_1 \in G$ the functions $a_n(g_1,-)$ and $b_n(g_1,-)$ belong to the ring $R(C_{g_1}(G)) \otimes \mathbb{Q}$, where $R(C_{g_1}(G))$ denotes the ring of complex characters of the group $C_{g_1}(G)$; see [16, Proposition 1.5]. Using the second and third conditions, and the usual scalar product of class functions on G [29, part 1, Section 2.3], we can see that $a_n(g_1,-)$ and $b_n(g_1,-)$ are indeed elements of $R(C_{g_1}(G))[1/|G|]$.

Remark 2.4. If $\vartheta \in \mathscr{E}\ell\ell_G^{-2k}$ and $\vartheta' \in \mathscr{E}\ell\ell_G^{-2k'}$, then $\vartheta\vartheta' \in \mathscr{E}\ell\ell_G^{-2(k+k')}$. Hence, the direct sum $\mathscr{E}\ell\ell_G^* = \bigoplus_{k\in\mathbb{Z}} \mathscr{E}\ell\ell_G^{-2k}$ has a natural structure of a graded ring.

2.2. The Green functor structure of $\mathscr{E}\ell\ell_G^*$

Let $\mathscr{G}$ be the category of finite G-sets. If S is an object of $\mathscr{G}$, then it has a decomposition

$$S = G/H_1 \sqcup G/H_2 \sqcup \cdots \sqcup G/H_n$$

into a disjoint union of orbits of G/H_i. We define a graded ring $\mathscr{E}\ell\ell_S^*$ by the equality

$$\mathscr{E}\ell\ell_S^* = \mathscr{E}\ell\ell_{H_1}^* \oplus \cdots \oplus \mathscr{E}\ell\ell_{H_n}^*,$$

where the ring structure is induced by coordinate-wise multiplication. Let H and K be two subgroups of G. If $H \subset K$ and $\vartheta \in \mathscr{E}\ell\ell_K^*$, then we define $\mathrm{rest}_H^K \vartheta \in \mathscr{E}\ell\ell_H^*$ as the restriction of ϑ to the subset $TH \times \mathfrak{h}_+$ of $TK \times \mathfrak{h}_+$. If H is a subgroup of G, then we shall write I_H for the kernel of rest_H^G.

Let $g \in G$, and let $c_g : H \to gHg^{-1} = {}^gH$ be the map defined by conjugation by g. Then c_g induces a map $\tilde{c}_g : TH \times \mathfrak{h}_+ \to T^gH \times \mathfrak{h}_+$. We define $c_g^* : \mathscr{E}\ell\ell_{^gH}^* \to \mathscr{E}\ell\ell_H^*$ by $\vartheta \to \vartheta\tilde{c}_g$. Finally, if $H \subset K$, we define $\mathrm{ind}_H^K : \mathscr{E}\ell\ell_H^* \to \mathscr{E}\ell\ell_K^*$ by the following formula:

$$(\mathrm{ind}_K^H \vartheta)((g_1, g_2), \tau) = \sum_{gH \in (K/H)[g_1, g_2]} \vartheta(g^{-1}g_1g,\ g^{-1}g_2g,\ \tau),$$

where $(g_1, g_2) \in TK$ and $(K/H)[g_1, g_2] = \{gH \in K/H \mid g_i gH \subset gH,\ i = 1, 2\}$.

The morphisms rest_H^K and ind_K^H admit canonical extensions to homomorphisms of groups $\mathrm{rest}_{S'}^S : \mathscr{E}\ell\ell_S^* \to \mathscr{E}\ell\ell_{S'}^*$ and $\mathrm{ind}_S^{S'} : \mathscr{E}\ell\ell_{S'}^* \to \mathscr{E}\ell\ell_S^*$ for any pair of finite G-sets S and S' such that $S' \subset S$. An analogous statement is true for the morphisms c_g^*. One can see that this family of morphisms induce a structure of a Green functor on $S \to \mathscr{E}\ell\ell_S^*$ [12, Section 3]; see for example [33, p. 275] for a definition of Green functors. A structure of Green functor is typical of the coefficient rings of multiplicative equivariant cohomology theories.

Among the Green functors there exists a universal object called the *Burnside ring*. The Burnside ring $A(H)$ of a finite group H is the Grothendieck ring of the monoid $\mathscr{H}$ of finite H-sets, where the addition is induced by the disjoint union of H-sets, and the product is induced by the product of H-sets. That the Burnside ring functor is universal among the Green functors means that given any Green functor $\mathbf{G}$, in particular $\mathscr{E}\ell\ell_G^*$, there exists a natural transformation of functors $A(-) \to \mathbf{G}$ [33, Proposition 8.12]. Let us recall that, as a consequence of [32, Proposition 1.2.3], the unit 1 of $A(G) \otimes \mathbb{Z}[1/|G|]$ can be written as an orthogonal sum of idempotents e_H, one for each conjugacy class of subgroups of G; therefore there exists a decomposition

$$A(G) \otimes \mathbb{Z}[1/|G|] = \bigoplus e_H A(G) \otimes \mathbb{Z}[1/|G|].$$

This decomposition induces a similar decomposition in any Green functor.

Lemma 2.5 (Devoto [12, Lemma 3.10]). *Let $e_H \in A(G)$ be an idempotent corresponding to the conjugacy class of a subgroup $H \subset G$. Then $e_H \mathscr{E}\ell\ell_G^* = 0$ unless $H = \langle g_1, g_2 \rangle$ for some pair of commuting elements $(g_1, g_2) \in TG$.*

Remark 2.6. The formula for the product $[G/H]\vartheta$, $\vartheta \in \mathscr{E}\ell\ell_G^*$ can be easily derived from [32, Proposition 6.2.3]. Lemma 2.5 follows from this formula and the explicit description of the idempotents e_H given in [3, 36].

Corollary 2.7. *Let $\mathscr{T}G$ be the category whose objects are the subgroups of G of the form $\langle g_1, g_2 \rangle$, where (g_1, g_2) is an element of TG, and whose morphisms are generated by the inclusions of groups and the conjugation by elements of G. Then*

(1) *The family of restrictions* $\mathcal{E}\ell\ell_G^* \to \mathcal{E}\ell\ell^*_{\langle g_1, g_2\rangle}$ *induce an isomorphism*

$$\mathcal{E}\ell\ell_G^* \to \varprojlim_{\langle g_1, g_2\rangle} \mathcal{E}\ell\ell^*_{\langle g_1, g_2\rangle}. \tag{2.6}$$

(2) *The family of induction morphisms induce an epimorphism*

$$\varinjlim_{\langle g_1, g_2\rangle} \mathcal{E}\ell\ell^*_{\langle g_1, g_2\rangle} \to \mathcal{E}\ell\ell_G^*. \tag{2.7}$$

(3) *If* $C(TG)$ *is a set of representatives of conjugacy classes in* $\mathcal{T}G$, *then*

$$\mathcal{E}\ell\ell_G^* \sim \bigoplus_{H \in C(TG)} \{\mathcal{E}\ell\ell_H^*\}^{W(H)}, \tag{2.8}$$

where $W(H)$ *is the Weyl group of* H.

Remark 2.8. Formula (2.6) follows directly from Lemma 2.5 and the theory of Green functors. This formula implies, by [32, Theorem 6.3.3], formula (2.7). Finally, the last formula follows from our Lemma 2.5; using the exact sequence 6.1.4, Proposition 6.1.6 and the formula 6.1.8 of [32]. See [32, Corollary 7.7.10] for a similar formula for the representation ring.

Remark 2.9. In formula (2.8) one should take in principle the localization of $\mathcal{E}\ell\ell_H$ at a certain subset $S(H)$ determined by H. This is not necessary since we proved in [12] that the elements of $S(H)$ are units of $\mathcal{E}\ell\ell_H^*$.

Remark 2.10. Let us remark that we can obtain the $W(H)$-invariant elements of $\mathcal{E}\ell\ell_H^*$ using the projector $p = (1/|N(H)|)\sum_{g \in N(H)} c_g^*$, where $N(H)$ is the normalizer of H in G.

2.3. The structure of $\mathcal{E}\ell\ell_G^*$

In this section we shall consider two problems.

Problem 1. We want to find generators of $\mathcal{E}\ell\ell_G^*$ considered as an algebra over $\mathcal{E}\ell\ell^*$.

Problem 2. We want to show that the functor $M \to M \otimes_{\mathcal{E}\ell\ell^*} \mathcal{E}\ell\ell_G^*$ from the category of graded modules over $\mathcal{E}\ell\ell^*$ to the category of graded modules over $\mathcal{E}\ell\ell_G^*$ is exact.

In order to solve both problems it suffices, by Corollary 2.7 and Remark 2.10, to consider the case $G = \langle g, h\rangle$ with $gh = hg$; hence in this section we shall always assume that G has this form.

As G is Abelian, then $TG = G \times G$ and $C_g(G) = G$, $\forall g \in G$. Let i be the homomorphism of groups from $(\mathbb{Z}/|G|\mathbb{Z})^*$ to $GL(2, \mathbb{Z}/|G|\mathbb{Z})$ defined by $i(n) = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Using this homomorphism we see that the actions of $(\mathbb{Z}/|G|\mathbb{Z})^*$ and $SL(2, \mathbb{Z})$ on TG are induced by the action σ of $GL(2,\mathbb{Z}/|G|\mathbb{Z})$ on TG given by

$$\begin{pmatrix} a & b \\ e & d \end{pmatrix} \times (g_1,g_2) \xrightarrow{\sigma} (g_1^d g_2^{-e}, g_1^{-b} g_2^a). \tag{2.9}$$

We shall fix a representative $[g_1,g_2]$ in each orbit $\overline{(g_1,g_2)}$ of σ and write $\Gamma([g_1,g_2])$ for the isotropy group of $[g_1,g_2]$ in $\Gamma_0(2)$. We shall denote the set of representatives $[g_1,g_2]$ by S.

Definition 2.11. The group $\mathscr{E}\ell\ell^{-2k}(\Gamma([g_1,g_2]))$ is the group of holomorphic functions $\vartheta : \mathfrak{h}_+ \to \mathbb{C}$ such that the following conditions hold:

(1) $\vartheta(\tau) = (e\tau+d)^{-k}\vartheta((a\tau+b)/(e\tau+d))$, for all $\left(\begin{smallmatrix} a & b \\ e & d \end{smallmatrix}\right) \in \Gamma([g_1,g_2])$;

(2) If τ_0 is any cusp of $\Gamma([g_1,g_2])$, and $\left(\begin{smallmatrix} a & b \\ e & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ is a matrix that transform the cusp $i\infty$ into the cusp τ_0, then the function $\vartheta'(\tau) = (e\tau+d)^k\vartheta(a\tau+b)/(e\tau+d)$ has a power series expansion at $i\infty$ of the form $\vartheta(\tau) = \sum_{n\geq m} a_n q^{2\pi i/|g_1|}$, with $a_n \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{|G|}, \exp 2\pi i/|G|\right]$.

We define $\mathscr{E}\ell\ell^*(\Gamma([g_1,g_2])) = \bigoplus_k \mathscr{E}\ell\ell^{-2k}(\Gamma([g_1,g_2]))$.

Let

$$\Lambda : \mathscr{E}\ell\ell_G^* \to \bigoplus_{[g_1,g_2]\in S} \mathscr{E}\ell\ell^*(\Gamma([g_1,g_2]))$$

be the ring homomorphism defined by

$$\Lambda(\vartheta) = \sum_{[g_1,g_2]\in S} \vartheta([g_1,g_2],-).$$

Remark 2.12. Using the transformation law of Definition 2.1 (1) we see that we can obtain the power series expansions of a function $\Lambda(\vartheta)$ at any cusp of $\Gamma([g_1,g_2])$ by considering the expansions at $i\infty$ of the functions $\vartheta((g_1,g_2),\tau)$ or $\vartheta'((g_1,g_2),\tau)$, where (g_1,g_2) are suitable elements of the orbit $\overline{(g_1,g_2)}$. From Definition 2.1 (2) it follows therefore that the function $\Lambda(\vartheta)$ belongs effectively to $\mathscr{E}\ell\ell^*(\Gamma([g_1,g_2])$.

Remark 2.13. The Galois action of $(\mathbb{Z}/|G|\mathbb{Z})^*$ on the rings $\mathscr{E}\ell\ell^*(\Gamma([g_1,g_2]))$ [22, Ch. 6, Section 3] induce an action $\sigma_{|G|}$ of $(\mathbb{Z}/|G|\mathbb{Z})^*$ on $\bigoplus_{[g_1,g_2]} \mathscr{E}\ell\ell^*(\Gamma([g_1,g_2]))$.

Proposition 2.14. *The morphism Λ is an isomorphism.*

Proof. We shall define an inverse Φ of Λ. Let $\Theta = \bigoplus \Theta_{[g_1,g_2]}$ be an element of $\bigoplus_{[g_1,g_2]\in S} \mathscr{E}\ell\ell^*(\Gamma([g_1,g_2]))$. If (h_1,h_2) is an element of TG, then there exists $[g_1,g_2] \in S$ and a matrix $\left(\begin{smallmatrix} m & r \\ k & j \end{smallmatrix}\right)$ in $GL(2,\mathbb{Z}/|G|\mathbb{Z})$ such that $\left(\begin{smallmatrix} m & r \\ k & j \end{smallmatrix}\right) \times [g_1,g_2] = (h_1,h_2)$. Let n be the determinant of $\left(\begin{smallmatrix} m & r \\ k & j \end{smallmatrix}\right)$. Then we can write

$$\begin{pmatrix} m & r \\ k & j \end{pmatrix} = i(n)p\left(\begin{pmatrix} a\,b \\ e\,d \end{pmatrix}\right),$$

where $\left(\begin{smallmatrix} a & b \\ e & d \end{smallmatrix}\right) \in \Gamma_0(2)$ and $p:\Gamma_0(2) \to GL(2, \mathbb{Z}/|G|\mathbb{Z})$ is the projection. We define

$$\Phi(\Theta)((h_1,h_2),\tau) = \sigma_{|G|}(i(n))\left((e\tau+d)^{-k}\Theta_{[g_1,g_2]}\left(\frac{a\tau+b}{e\tau+d}\right)\right).$$

It is not difficult to check that Φ is well defined and that it is an inverse of Λ. From the integrality condition (2) in Definition 2.11 it follows that $\Phi(\Theta)$ satisfies condition (2) in Definition 2.1.

The rings $\mathcal{E}\ell\ell(\Gamma([g_1,g_2]))$, as the rings of classical modular forms of higher level, have a modular interpretation related to elliptic curves; see, for example, [22] for the classical case. The main difference is that, due to the integrality condition (condition (2) in Definition 2.11) in the coefficients of the expansions of the elements of $\mathcal{E}\ell\ell^*(\Gamma([g_1,g_2]))$, one has to work with *elliptic curves defined over general schemes* and with the *arithmetic moduli* of elliptic curves; see [11, Introduction]. We shall recall briefly some relevant definitions and results from [11, 19].

Definition 2.15. Let S be a scheme. Then an *elliptic curve* $\boldsymbol{E}: E \xrightarrow{p} S$ over S is a proper and flat morphism of relative dimension at most one and constant Euler–Poincaré characteristic 0, together with a section $s:S \to E$. We shall also write $E|S$ for an elliptic curve $E \xrightarrow{p} S$.

We shall write $\Omega_{E|S} \xrightarrow{\pi} E$ for the invertible sheaf of relative differentials, and define $\omega_{E|S} = p_*(\Omega_{E|S})$.

An elliptic curve admits a unique structure of group scheme such that the section s is the identity element. Let $[n]:E \to E$, for $n \in \mathbb{N}$, be the map induced by multiplication by n in the group scheme structure on the elliptic curve. Then, if n is invertible in S, the map $[n]$ is étale. We shall denote the kernel $\ker[n]$ by $E[n]$.

Definition 2.16. Let A be an Abelian group. An *A-structure* on an elliptic curve $E \to S$ is a morphism of abstract groups $\phi: A \to E$ such that the effective Cartier divisor D_A of degree $\#A$ defined by

$$D_A = \sum_{a \in A} [\phi(a)]$$

is a subgroup of $E|S$.

Let **Ell** be the category whose objects are the elliptic curves $E \xrightarrow{p} S$ and whose morphisms are the commutative squares

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ S \arrow[r] & S' \end{tikzcd}$$

such that $E = S \times_{S'} E'$. A *moduli problem* $\mathcal{M}$ is a contravariant functor $\mathcal{M}$ from **Ell** to the category **Sets** of sets. A moduli problem $\mathcal{M}$ is called *representable* if and only if there exists an elliptic curve $E_{\mathcal{M}} \to S_{\mathcal{M}}$ and a natural isomorphism of functors $\Phi : \mathcal{M} \to [-, E_{\mathcal{M}} \to S_{\mathcal{M}}]_{\mathrm{Ell}}$. If $\mathcal{M}$ and $\mathcal{N}$ are two moduli problems, then the *simultaneous moduli problem* $\mathcal{M} \times \mathcal{N}$ is the functor defined by $\mathcal{M} \times \mathcal{N}(E|S) = \mathcal{M}(E|S) \times \mathcal{N}(E|S)$.

Example 2.17 (*A-structures*). Let A be an Abelian group. Then the moduli problem of *A-structures* $\mathcal{M}_A$ is the functor

$$E \to \{\Phi : A \to E | \Phi \text{ is an } A\text{-structure}\}.$$

Example 2.18 (*$\Gamma_0(n)$-structures*). The moduli problem of $\Gamma_0(n)$-structures is the set of isogenies $\alpha : E \to E'$ of degree n such that locally f.p.p.f. (faithfully flat of finite presentation) $\ker \alpha$ admits a generator.

Example 2.19 (*Jacobi structures*). The moduli problem of *Jacobi structures* is the functor $\mathcal{M}_J$ that assigns to each elliptic curve $E \to S$ the set of pairs (α, ω), with α a $\Gamma_0(2)$ structure on $E|S$, and ω an $\mathcal{O}_S$ basis of $\omega_{E|S}$.

Definition 2.20. A *modular form f of level A and weight k* is a rule that assigns to each triple $(E|\mathrm{spec}(R), \phi, \omega)$ formed by an elliptic curve $E|\mathrm{spec}(R)$ over the spectrum of a ring R together with an A-structure ϕ on E and a basis ω of $\omega_{E|\mathrm{spec}(R)}$ an element of R in such a way that the following conditions are satisfied:

(1) The element $f(E|\mathrm{spec}(R), \phi, \omega)$ depends only on the R-isomorphism class of the triple $(E|\mathrm{spec}(R), \phi, \omega)$.

(2) If λ is a unit of R, then $f(E|\mathrm{spec}(R), \phi, \lambda\omega) = \lambda^{-k} f(E|\mathrm{spec}(R), \phi, \omega)$.

(3) The formation of f commutes with arbitrary extensions of scalars.

We shall restrict our attention to elliptic curves $E \to S$ defined over schemes where 2 is invertible. Since elliptic cohomology is defined over $\mathbb{Z}[\frac{1}{2}]$ we do not lose any generality.

Proposition 2.21. *The pair formed by the universal Jacobi quartic E_J of equation*

$$Y^2 = 1 - 2\delta X^2 + \varepsilon X^4$$

defined over $\mathbb{Z}[1/2, \delta, \varepsilon, \Delta^{-1}]$ and $\omega = dX/Y$ represents the moduli problem of Jacobi structures. We shall write S_J for the spectrum of $\mathbb{Z}[\frac{1}{2}, \delta, \varepsilon, \Delta^{-1}]$.

Remark 2.22. The proof of this proposition is similar to the proof of [13, Proposition 2]; this proof deals with the case $S = \mathrm{spec}\, k$, where k is a field of characteristic different from 2 but it can be easily modified, using the techniques of [19, Ch. 2], to cover the general case.

We shall be interested in the simultaneous moduli problems $\mathcal{M}_{A,J} = \mathcal{M}_A \times \mathcal{M}_J$, where $A = \langle g_1, g_2 \rangle$ for some pair $(g_1, g_2) \in TG$. For simplicity we shall restrict the discussion to the case $A = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. The general case can be obtained using the results of

[19, Ch. 7]. Let $\mathcal{M}_J(n)$ be the affine subscheme of $E_J[n]\times_{S_J} E_J[n]$ consisting of pairs of points (P,Q) that form a basis of $E_J[n]$. Let

$$E_{n,J}=E_J\times_{S_J}\mathcal{M}_J(n)\to\mathcal{M}_J(n)$$

be the elliptic curve obtained from E_J by change of basis; note that $\mathcal{M}_J(n)$ has a natural structure of scheme over S_J. The curve $E_{n,J}|\mathcal{M}_J(n)$ has a canonical Jacobi structure (α,ω) induced by the Jacobi structure of E_J and the canonical $\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$-structure β induced from $\mathcal{M}_J(n)$. From Lemma 3.6, the result 4.2, and Theorem 5.1.1 of [19] it follows that $(E_{n,J}|\mathcal{M}_J(n),\alpha,\beta,\omega)$ represents $\mathcal{M}_{\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z},J}$.

If f is a modular form of level $(\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z},J)$, then f is completely characterized by its value $f(E_{n,J}|\mathcal{M}_J(n),\alpha,\beta,\omega)\in\mathcal{O}(\mathcal{M}_J(n))$; therefore we have an inclusion

$$\mathcal{E}\ell\ell^*(\Gamma(n))\subset\mathcal{O}(\mathcal{M}_J(n)). \tag{2.10}$$

Let us describe the scheme $\mathcal{M}_J(n)$ explicitly. The multiplication by n in E_J is described, see [17], by

$$[n]X=X^{n^2}F_n(X^{-1})F_n^{-1}(X), \tag{2.11}$$

$$[n]Y=YG_n(X)F_n^{-2}(X), \tag{2.12}$$

for certain polynomials $F_n,G_n\in\mathcal{E}\ell\ell^*[X]$; we shall write $T_n(X)=X^{n^2}F_n(X^{-1})$. Therefore

$$\mathcal{O}(E_J[n])=\mathcal{E}\ell\ell^*[X,Y]/(Y^2-1+2\delta X^2-\varepsilon X^4,T_n(X),G_n(X)Y=F_n(X)^2).$$

We shall write $\mathcal{O}(E_J[n])=\mathcal{E}\ell\ell^*[x,y]$ and $\mathcal{O}(E_J[n]\times_{S_J}E_J[n])=\mathcal{E}\ell\ell^*[x_1,y_1,x_2,y_2]$. For each pair $(a,b)\in\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$ such that $(a,b)\neq 0$ we have an element $S_{(a,b)}\in\mathcal{O}(\mathcal{M}_J(n))$ defined by

$$S_{(a,b)}(P,Q)=x(aP+bQ), \tag{2.13}$$

where $aP+bQ\in E_J[n]$ is obtained using the group structure of $E_J[n]$ and x is the restriction of the X-coordinate of the universal Jacobi quartic. A pair (P,Q) is in $\mathcal{M}_J(n)$ if and only if $S_{(a,b)}(P,Q)\neq 0$ for all the pairs (a,b); hence

$$\mathcal{O}(\mathcal{M}_J(n))=\mathcal{O}(E_J[n]\times_{S_J}E_J[n])[S_{(a,b)}^{-1}].$$

It is easy to see that $x_1=S_{(1,0)}$ and $x_2=S_{(0,1)}$. Using the addition law for the Jacobi quartic [17, 23] one can easily see that

$$y_1=\frac{1}{2}\left[\frac{(1-\varepsilon^2S_{(1,0)}^4)S_{(2,0)}}{S_{(1,0)}}\right]\quad\text{and}\quad y_2=\frac{1}{2}\left[\frac{(1-\varepsilon^2S_{(0,1)}^4)S_{(0,2)}}{S_{(0,1)}}\right];$$

hence $\mathcal{O}(\mathcal{M}_J(n))=\mathcal{E}\ell\ell^*[S_{(a,b)},S_{(a,b)}^{-1}]$. We shall see that the elements $S_{(a,b)}$ and their inverses are modular forms of level $\Gamma_0(2)$ and weight 2. Therefore the inclusion (2.10) is really an equality.

Remark 2.23. If E is any Jacobi quartic over any ring R where n is invertible, then we can always define "functions" $S^E_{(a,b)}$ as in (2.12).

Remark 2.24. If E is defined over a field k, then $k(S_{(a,b)})$ is isomorphic to the extension $k(x)$ obtained by adjoining the points of order n of E

We can now discuss an analytical interpretation of the elements $S_{(a,b)}$. Let $s(u,\tau)$ be the function defined by

$$(s(u,\tau)) = \frac{1}{2\sinh(u/2)} \prod_{n=1}^{\infty} \left[\frac{(1-q^n)^2}{(1-q^n e^u)(1-q^n e^{-u})} \right]^{-1^n}. \tag{2.14}$$

Then the functions $s(u,\tau)$ and $s'(u,\tau)$ $(\partial/\partial u)(s(u,\tau))$ parametrize the Jacobi quartic

$$y^2 = 1 - 2\delta x^2 + \varepsilon x^4.$$

where $\delta(\tau)$, $\varepsilon(\tau)$ are modular forms for the group $\Gamma_0(2)$. Let c be a natural number bigger than 2. Then we shall call

$$s_{(a,b)}(\tau) = s\left(4\pi i\left(\frac{a\tau}{c} + \frac{b}{c}, \tau\right), \tau\right). \tag{2.15}$$

The functions $s_{(a,b)}(\tau)$ are the analytical version of the algebraically defined $S_{(a,b)}$; see the last pages of [14] where one can also see the modular properties of these functions.

Proposition 2.25. *The ring $\mathscr{E}\ell\ell^*_G$ is a flat $\mathscr{E}\ell\ell^*$-module.*

Proof. The moduli problems $\Gamma(g_1, g_2), J$ are flat (this is due to the fact that the problem $\Gamma(n)$, $n \geq 3$ is flat [19]). Therefore $\mathscr{E}\ell\ell^*(\Gamma(g_1, g_2))$ is flat over $\mathscr{E}\ell\ell^*$. The proposition follows from Corollary 2.7 and the existence of the morphism Λ. We refer to [7] for another proof of the fact that rings of modular forms of higher level are flat. $\square$

2.4. The ideals of $\mathscr{E}\ell\ell^*_G$

The groups $H \subset G$ generated by a pair of elements g, h such that $gh = hg$ play a role in equivariant elliptic cohomology similar to the role played by the cyclic groups in equivariant K-theory.

Definition 2.26. We shall write $\mathscr{T}G$ for the family of subgroups of H of G such that there exists an epimorphism $\mathbb{Z} \times \mathbb{Z} \to H$.

Let P be an homogeneous prime ideal of $\mathscr{E}\ell\ell^*_G$. Then we shall say that a subgroup H of G is *the support* of P if the following conditions are satisfied.

(1) There exist an homogeneous prime ideal P' of $\mathscr{E}\ell\ell^*_H$ such that

$$P = (\mathrm{rest}^G_H)^{-1}(P').$$

In this case we shall say that P *comes from* H.

(2) If $H' \in H$ is any subgroup, then $P \neq (\mathrm{rest}^G_{H'})^{-1}(P'')$ for any homogeneous prime ideal P'' of $\mathscr{E}\ell\ell^*_{H'}$.

The support of an homogeneous prime ideal is defined up to conjugation.

Proposition 2.27. *The support of any homogeneous prime ideal $P \in \mathscr{E}\ell\ell^*_G$ is the conjugacy class of a subgroup $H \in \mathscr{T}G$.*

This result follows from (2.8). The following corollary is a general fact. The proof is basically the same as the proof of [27, Proposition 3.7].

Corollary 2.28. *Let P be an homogeneous prime ideal of $\mathscr{E}\ell\ell^*_G$ and let H be a subgroup of G. Then the following statements are equivalent*:

(1) *P comes from $\mathscr{E}\ell\ell^*_H$ via the restriction $\mathscr{E}\ell\ell^*_G \to \mathscr{E}\ell\ell^*_H$.*

(2) *The kernel* $\ker(\mathscr{E}\ell\ell^*_G \to \mathscr{E}\ell\ell^*_H)$ *is contained in P.*

(3) *The localized module $\{\mathscr{E}\ell\ell^*_H\}_P \neq 0$.*

Corollary 2.29. *Let H be a subgroup of G. If K is in the support of an ideal P of $\mathscr{E}\ell\ell^*_H$, then K is in the support of $(r^G_H)^{-1}(P)$.*

3. Equivariant elliptic cohomology

3.1. The geometric twisted elliptic genus

Recall that the *universal elliptic genus* $\Phi : \mathrm{MSO}_* \to \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ can be defined using a K-theoretical characteristic class, called *Witten's* characteristic class,

$$\Theta : KO^* \to K^*[[q]]. \tag{3.1}$$

The notation in (3.1) is the following: we write KO^* for real K-theory and K^* for complex K-theory, q is a formal variable, and $K^*[[q]]$ is the functor that assigns to each space X homotopy equivalent to a finite CW-complex X the ring of formal power series in q with coefficients in $K^*(X)$; see [12, 24, 26] for a precise definition of Witten's characteristic class, and [21, 28] for references about the elliptic genus. When X is a spin manifold, the elliptic genus evaluated in the bordism class defined by X has a geometric interpretation in terms of S^1-equivariant operators on the space of free loops on X [28, 35]. This interpretation is related to the theory of non-linear sigma models [34].

We will define our equivariant version of the elliptic genus using a twisted generalization of Witten's characteristic class. The definition of this class is motivated by the theory of orbifold sigma models. Our "twisted" version of the functor $K^*[[q]]$ is the functor $\mathscr{K}_G : G$-spaces $\to$ Rings given by

$$\mathscr{K}^*_G(X) = \bigoplus_{(g_1, g_2) \in TG} \{K^*(X^{g_1, g_2}) \otimes_{\mathbb{Z}} R(\langle g_2 \rangle)\}[[q^{1/|g_1|}]], \tag{3.2}$$

where X is a compact G-space, $X^{g_1,g_2}=\{x\in X \mid g_1x=g_2x=x\}$, and $R(\langle g_2\rangle)$ is the ring of complex characters of the group generated by g_2. It is not difficult to show that $\mathscr{K}_G$ is, in the sense of [32, Definition 6.7], a G-equivariant cohomology theory.

Let X be a compact G-space, and let $E\to X$ be a G-equivariant complex vector bundle. Then, for each pair $(g_1,g_2)\in TG$, the restriction $E|_{X^{g_1,g_2}}\to X^{g_1,g_2}$ admits a decomposition

$$E|_{X^{g_1,g_2}}=\bigoplus_{-|g_1|/2<j<|g_1|/2}\left\{\bigoplus_{-|g_2|/2<k<|g_2|/2} F_{jk}\right\},\tag{3.3}$$

where the $\langle g_1,g_2\rangle$-equivariant complex vector bundles F_{jk} are characterized by the fact that g_1 acts fiberwise as $\exp\{2\pi ij/|g_1|\}$ and g_2 as $\exp\{2\pi ik/|g_2|\}$. We define

$$\begin{aligned}\bar\theta_G(E|_{X^{g_1,g_2}}) = \bigotimes_{\substack{-|g_1|/2<j<|g_1|/2\\ -|g_2|/2<k<|g_2|/2}} &\left[\bigotimes_{s\geq 1}(\wedge_{[w^{2k/c'}][-q^{2s-1}]}[F_{jk}])\right.\\ &\left.\otimes\bigotimes_{s\geq 0}(S_{[w^{2k/c'}][q^{2s}]}[F_{jk}])\right].\end{aligned}\tag{3.4}$$

In (3.4) we are taking $c=|g_1|$, $c'=|g_2|$, $s=(nc+j)/c$ with $n\in\mathbb{Z}$, and $R(\langle g_2\rangle)=\mathbb{Z}[w]$. If E is a real G-equivariant vector bundle, then we define

$$\theta_G(E|_{X^{g_1,g_2}})=\bar\theta_G((E\otimes\mathbb{C})|_{X^{g_1,g_2}}).\tag{3.5}$$

The conventions used in the decomposition of $(E\otimes\mathbb{C})|_{X^{g_1,g_2}}$ are the usual ones in index theory; see [6]. In [12] we showed that θ_G has an extension to a G-equivariant stable exponential class, which we called *Witten's twisted class*,

$$\theta_G: KO_G^*\to\mathscr{K}_G^*.$$

Let X be a closed, oriented, compact, Riemannian manifold of dimension $2k$ where G acts by orientation-preserving isometries. We shall assume, just to simplify the formulae, that each X^{g_1,g_2} is connected. This is a minor assumption that can easily be removed. As the order $|G|$ of G is odd, the orientation on X induces an orientation on each one of the submanifolds X^{g_1,g_2} [6, p. 584]. Recall that, since *BSpin* and *BSO* are homotopically equivalent at odd primes [31, p. 336], orientable manifolds are orientable for $K^*\otimes\mathbb{Z}[\frac{1}{2}]$-theory. Therefore, for each pair $(g_1,g_2)\in TG$, there exists a Gysin map

$$\pi_!^{g_1,g_2}: K^*(X^{g_1,g_2})\otimes\mathbb{Z}\left[\frac{1}{2}\right]\to K^*(pt)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$$

induced by the projection $\pi: X^{g_1,g_2}\to pt$. The family of maps $\pi_!^{g_1,g_2}$ induces a Gysin map $\pi_!:\mathscr{K}_G^*(X)\to\mathscr{K}_G^*$.

Definition 3.1. The *geometric twisted elliptic genus* Φ_G is defined by the equality

$$\Phi_G(X) = \pi_! \left(\frac{\theta_G([TX])}{\theta_G([\dim(TX)])} \right) = \sum_{(g_1,g_2)\in TG} \Phi(X^{g_1,g_2}) \in \mathscr{K}_G(pt),$$

where $[\dim(TX)]$ is the element of $K_G(X)$ that we obtain if we replace all the bundles F_{jk} in formula (3.3) by topologically trivial bundles T_{jk}, where $\dim_{\mathbb{C}} T_{jk} = \dim_{\mathbb{C}} F_{jk}$ and where g_1 and g_2 act in the same way as in F_{jk}.

Remark 3.2. Besides the class Θ, Witten considers in [35] two characteristic classes related to Θ by elements of $SL(2,\mathbb{Z})$ not in $\Gamma_0(2)$. We shall be interested in one of these classes, which we shall denote by $\overline{\theta}'$, that is related to the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We shall write Φ' for the genus associated to it. This genus has, as the elliptic genus, a natural geometric interpretation as the S^1-equivariant index of some operator of Dirac type on loop spaces. The study of the equivariant index of this operator on twisted loop space leads us to two objects: an equivariant generalization $\overline{\theta}'_G$ of $\overline{\theta}'$ and a "new" twisted version Φ'_G of the elliptic genus. For simplicity we shall only give here the contribution of the bundles F_{jk} to $\Phi'_G(X)$. Using the splitting principle it suffices to consider the case $\dim_{\mathbb{C}} F_{jk} = 1$, in this case the contribution of F_{jk} to $\Phi'_G(X)$ is

$$\frac{\left[\sum_{k\in\mathbb{N}} q^{1/2(m+1/2-k/c')^2} \mu^{(m+1/2-k/c')} F_{jk}^{(2c'm+c'/2-2k)}\right]}{\left[\sum_{k\in\mathbb{N}} q^{1/2(m+1/2-k/c')^2} \mu^{(m+1/2-k/c')} e^{\pi i(m+1/2-k/c')} F_{jk}^{(2c'm+c'/2-2k)}\right]^{(-1)}}, \tag{3.6}$$

where $\mu = \exp 2\pi i j/c$. The series involved in (3.6) are related to the power series expansions of theta functions with characteristics.

It is easy to show that Φ_G induces a ring homomorphism $\Omega_*^G \to \mathscr{K}_G(pt)$, where Ω_*^G is the geometric oriented equivariant bordism of [8]. As we are interested in cohomology we shall suppose that Φ_G it is defined on $\Omega_G^* = \Omega_{-*}^G$.

Pick $\tau \in \mathfrak{h}_+$ and let $q = \exp\{2\pi i\tau\}$. We define $\Phi_G(X)((g_1,g_2),\tau)$ as the evaluation of $\Phi_G(X^{g_1,g_2})$ at g_2 and τ. The evaluation at g_2 is done via the identification between representations and characters.

Proposition 3.3. *The function* $((g_1,g_2),\tau) \to \Phi_G(X)((g_1,g_2),\tau)$ *belongs to the ring* $\mathscr{E}\ell\ell_G^*$.

Proposition 3.4. *The twisted elliptic genus defines a graded ring homomorphism* $\Phi_G : \Omega_G^* \to \mathscr{E}\ell\ell_G^*$.

Remark 3.5. The proof of both propositions can be done using a cohomological formula, obtained using the Pontrjagin character, for the twisted elliptic genus; see [12, Section 2] for the details. We still have to check that if $X \in \Omega_G^*$, then the function

$\Phi_G(X)$ satisfy all the conditions of Definition (2.1). This follows from the cohomological formula for the twisted elliptic genus, the transformation laws for theta functions with characteristics which can be found in [12, Section 2] or [17], and the fact that the expansion of the functions $\Phi_G(X)'(g_1, g_2\tau)$ correspond to the series $\Phi'_G(X)$, where Φ'_G is the genus of Remark 3.2.

3.2. Homotopy-theoretic equivariant oriented bordism

Let (X,A) be a pair of G-spaces. Using cellular approximation we can suppose that X is a G-CW complex and that A is a G-CW sub-complex. Then, for each real orthogonal representation V of G of dimension $|V|$, there exists a suspension homomorphism

$$\sigma(V) : \Omega_n^G(X,A) \to \Omega_{n+|V|}^G\ (D(V)\times X, (D(V)\times A)\cup(S(V)\times X)), \tag{3.7}$$

where $D(V)$ (respectively $S(V)$) is the unit disk (respectively the unit sphere) in V. If $(M,\partial M)\to(X,A)$ is a representative of a bordism class $[X]\in\Omega_n^G(X,A)$, then $\sigma(V)([X])$ is the bordism class of

$$(D(V)\times M, \partial(D(V)\times M))\to(D(V)\times X,(D(V)\times A)\cup(S(V)\times X)).$$

Remark 3.6. If V and W are two finite-dimensional real orthogonal representations of G, and $V\cap W=0$, then $\sigma(V\oplus W)=\sigma(V)\sigma(W)$.

Remark 3.7. If V is a non-trivial representation, then the suspension $\sigma(V)$, is not, in general, an isomorphism.

Let $\mathcal{U}$ be an orthogonal representation of G that contains an infinite number of times each finite dimensional representation of G. We shall write $F\mathcal{U}$ for the set of finite dimensional G sub-spaces of $\mathcal{U}$. We define an order $<$ on $F\mathcal{U}$ by: $V<W$ if V is isomorphic to some G-submodule of W. Using this order, and Remark 3.6, we see that $\{\Omega_*^G(X\times D(V),(D(V)\times A)\cup(S(V)\times X))\}$ is a direct system of graded groups indexed by the ordered set $F\mathcal{U}$.

Definition 3.8. Let (X,A) be a pair of G-CW complexes, $A\subset X$. The *homotopy theoretic equivariant oriented bordism group* $MSO_*^G(X,A)$ [10, p. 72] of the pair (X,A) is the graded group defined by the equality

$$MSO_n^G(X,A)=\varinjlim_{V\in F\mathcal{U}}\Omega_{n+|V|}^G(D(V)\times X,(D(V)\times A)\cup(S(V)\times X)). \tag{3.8}$$

Remark 3.9. The way in which the theory $MSO_*^G(X,A)$ has been defined corresponds to the definition of [10] only when the order of the group is odd. The reason is that, for $|G|$ odd, the universal equivariant orientation in the sense of [10] is completely determined by an orientation preserving action of G [9, Section 6].

3.3. The homotopy theoretic twisted elliptic genus

Using the explicit description of MSO^G_* given by (3.8) we see that in order to extend the domain of definition of Φ_G to $MSO^*_G = MSO^G_{-*}$ it suffices to define, for each $V \in F\mathscr{U}$, a morphism

$$\Phi^V_G : \Omega^G_{n+|V|}(D(V), S(V)) = \tilde{\Omega}^G_{n+|V|}(\Sigma(V)) \to \mathscr{E}\ell\ell^G_n,$$

where $\Sigma(V) = D(V)/S(V)$, in a way compatible with the suspension maps (3.7).

Let us suppose that $G = \langle g, h\rangle$ with $gh = hg$. Let $(M, \partial M) \to (D(V), S(V))$ be a representative of a bordism class $[X]$ in $\Omega^G_*(D(V), S(V))$. In the definition of $\Phi^V_G(M, \partial M)$ (g_1, g_2, τ), where (g_1, g_2) is an element of TG, we have to consider two cases.

Case 1: Suppose that $G = \langle g_1, g_2\rangle$. Then V admits a decomposition

$$V = V_0 \oplus (\textstyle\bigoplus_{jk} n_{jk} V_{jk}),$$

where $V_0 = \{v \in V \mid gv = v, \forall g \in G\}$, V_{jk} are the non-trivial irreducible representations of G and n_{jk} is the multiplicity with which the representation V_{jk} appears in V. We shall write $V_1 = \bigoplus_{jk} n_{jk} V_{jk}$.

The suspension $\sigma(V_0) : \Omega^G(D(V_1), S(V_1)) \to \Omega^G(D(V), S(V))$ is an isomorphism. Suppose that $(N, \partial N) \xrightarrow{p} (D(V_1), S(V_1))$ represents the class $\sigma^{-1}(V_0)([X])$. By hypothesis $D(V_1)^{g_1, g_2} = 0$ so $N^{g_1, g_2} \subset p^{-1}(0)$. Let $TN_{N^{g_1, g_2}}$ be the restriction of the tangent bundle of N to N^{g_1, g_2}. Then we have a decomposition $TN_{N^{g_1, g_2}} = TF \oplus NF$ of $TN_{N^{g_1, g_2}}$ into the part TF tangent to the fiber of $p : N \to D(V)$ and the normal part NF. Then we define

$$\Phi^V_G(M, \partial M)(g_1, g_2, \tau) = \prod_{jk} s_{jk}^{-n_{jk}}(\tau) \left\langle \frac{\Phi_G([TF])}{\Phi_G([\dim TF])}, [N^{g_1, g_2}] \right\rangle. \tag{3.9}$$

The conventions in (3.9) are the same that we used in Definition 3.1. The functions $s_{jk}(\tau)$ are the functions defined in (2.15).

Case 2: Suppose now that $H = \langle g_1, g_2\rangle \neq G$ and let V' and $(M', \partial M')$ be the representation V and the manifold M with the H action. Then we define

$$\Phi^V_G(M, \partial M)(g_1, g_2, \tau) = \Phi^{V'}_H(M', \partial M')(g_1, g_2, \tau), \tag{3.10}$$

where the right-hand side is defined as in case 1.

It is not difficult to see that the family of morphisms Φ^V_G induces an extension Φ_G of the geometric twisted elliptic genus. If G is now any finite group of odd order, then we define $\Phi_G : MSO^G_* \to \mathscr{E}\ell\ell^G_*$ as the composition

$$MSO^G_* \xrightarrow{r} \varprojlim_{H \in \mathscr{T}G} MSO^H_* * \xrightarrow{\Phi} \varprojlim_{H \in \mathscr{T}G} \mathscr{E}\ell\ell^H_* \xleftarrow{\text{rest}^{-1}} \mathscr{E}\ell\ell^G_*, \tag{3.11}$$

where r is induced by the restriction morphisms $r^G_H : MSO^G_* \to MSO^H_*$, Φ is induced by the family of morphisms Φ_H, and rest^{-1} is the inverse of the homomorphism defined in (2.6).

Let $\Delta_{MSO} = [\mathbb{HP}^2]([\mathbb{CP}^2]^2 - [\mathbb{HP}^2])^2 \in MSO_* \subset MSO_*^G$. Then $\Phi_G(\Delta_{MSO}) = \Delta \in \mathscr{E}\ell\ell_*^G$. As Δ is invertible in $\mathscr{E}\ell\ell_*^G$ the twisted elliptic genus admits a factorization

$$\begin{array}{ccc} MSO_*^G & \xrightarrow{\Phi_G} & \mathscr{E}\ell\ell_*^G \\ \Big\downarrow & & \Big\| \\ MSO_*^G[\frac{1}{\Delta_{MSO}}] & \xrightarrow{\Phi_G} & \mathscr{E}\ell\ell_*^G. \end{array}$$

We define $mso_G^*(X,A) = MSO_G^*[1/|G|, 1/\Delta_{MSO}](X,A)$. It is easy to see that mso_G^* is a G-equivariant stable multiplicative cohomology theory.

Proposition 3.10. *The homotopy theoretic twisted elliptic genus* $\Phi_G : mso_G^* \to \mathscr{E}\ell\ell_G^*$ *is a transformation of Green functors.*

Proposition 3.11. *The homotopy theoretic twisted elliptic genus* $\Phi_G : mso_G^* \to \mathscr{E}\ell\ell_G^*$ *is an epimorphism.*

Proof. Using Proposition 3.10 and Corollary 2.7 we see that it suffices to consider the case $G = \langle g_1, g_2 \rangle$. In this case it suffices to prove, using the isomorphism Λ, that if $\langle g_1, g_2 \rangle = G$, then $\forall \theta \in \mathscr{E}\ell\ell^*(\Gamma(g_1, g_2))$ there exists $[M] \in mso_G^*$ such that $\Phi_G([M]) = \theta$.

By the structure theorem for Abelian groups we can suppose that $G = \mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/c'\mathbb{Z}$ where c'/c. We shall discuss the case $c' = c$ and refer the reader to [12] for the general case. In this case

$$\mathscr{E}\ell\ell^*(\Gamma_0(c) \cap \Gamma_0(2)) = \mathscr{E}\ell\ell^*[s_{(a,b)}, s_{(a,b)}^{-1}(\tau)], \tag{3.12}$$

where $s_{(a,b)}(\tau)$ are the functions defined in (2.15). The functions $s_{(a,b)}(\tau)$ can be obtained applying the homotopy twisted elliptic genus to the Euler class of the irreducible representation $V_{(a,b)}$ of weight (a,b) of $\mathbb{Z}_c \times \mathbb{Z}_c$. Applying formula (3.9) to suitable elements of MSO_G^* we can see that the elements $s_{(a,b)}^{-1}(\tau)$ are also in the image of the twisted genus. □

3.4. Equivariant elliptic cohomology

We shall describe now the results of Section 5 of [12].

Definition 3.12. Let (X,A), $A \subset X$ be a pair of G-spaces formed by a finite G-CW complex X and a sub-complex A. Then the *equivariant elliptic cohomology* $\mathscr{E}\ell\ell_G^*(X,A)$ of the pair (X,A) is the graded tensor product

$$\mathscr{E}\ell\ell_G^*(X,A) = mso_G^*(X,A) \bigotimes_{mso_G^*} \mathscr{E}\ell\ell_G^*, \tag{3.13}$$

where $\mathscr{E}\ell\ell_G^*$ is considered as a graded algebra over mso_G^* via the twisted elliptic genus.

Remark 3.13. It is easy to show, by a simple argument of change of rings, that $\mathscr{E}\ell\ell_G^*(X,A) \sim MSO_G^*(X,A) \otimes_{MSO_G^*} \mathscr{E}\ell\ell_G^*$.

In [12, Section 5] we showed that the functor $(X,A) \to \mathscr{E}\ell\ell_G^*(X,A)$ defines a G-equivariant cohomology theory. The main point in the proof was to show that the Green functor structures of $H \to mso_H^*(X,A)$ and $H \to \mathscr{E}\ell\ell_H^*$ induces a Green functor structure on $H \to \mathscr{E}\ell\ell_H^*(X,A)$. Let us quote the relevant results.

Proposition 3.14. *Let H be a subgroup of G and let I_H be the kernel of the homotopy theoretic twisted elliptic genus*

$$\Phi_H : mso_H^* \to \mathscr{E}\ell\ell_H^*.$$

Then $I_H = \mathrm{restr}_H^G(I_G) mso_H^*$.

Proposition 3.15. *Let X be a finite G-CW complex. Then the functor $H \to \mathscr{E}\ell\ell_H^*(X)$ has a natural structure of Green functor.*

It is straightforward to check that the restriction and conjugation morphisms of the cobordism functor pass to equivariant elliptic cohomology. Proposition 3.14 implies that also the induction functors pass to elliptic cohomology.

Since $H \to \mathscr{E}\ell\ell_H^*(X,A)$ is a Green functor defined over $\mathbb{Z}[1/|G|]$ we can decompose it using the idempotents e_H of the Burnside ring $A(G)$. As the Green structure of $\mathscr{E}\ell\ell_{(-)}^*(X,A)$ is obtained from the Green functor structure of cobordism by passing to the quotient, then $e_H(a \otimes b) = e_H(a) \otimes e_H(b)$, for any pair $a \in mso_G^*(X,A)$, and $b \in \mathscr{E}\ell\ell_G^*$. The products $e_H \mathscr{E}\ell\ell_G^*$ are described in Lemma 2.5 and Corollary 2.7. A description of the products $e_H mso_G^*(X,A)$ can be obtained using [15, Lemma 2.2; 20, Lemma 4.7]. Combining both descriptions we obtain the following theorem.

Theorem 3.16. *There exists a natural equivalence of functors*

$$\mathscr{E}\ell\ell_G^*(X) \to \bigoplus_{\langle g_1,g_2\rangle \in \mathscr{C}\mathscr{C}} [\mathscr{E}\ell\ell^*(X^{g_1,g_2}) \otimes_{\mathscr{E}\ell\ell^*} \mathscr{E}\ell\ell_{\langle g_1,g_2\rangle}^*]_{S(\langle g_1,g_2\rangle)}^{W\langle g_1,g_2\rangle}. \tag{3.14}$$

The sum in (3.14) *is being taken over a complete set of representatives of conjugacy classes of subgroups of the form $\langle g_1, g_2\rangle$ and we localize with respect to the set $S(\langle g_1,g_2\rangle)$ which is the image of the ideal $q(\langle g_1,g_2\rangle,0) = \ker e_H$ under the natural homomorphism $A(G) \to \mathscr{E}\ell\ell_G^*$.*

Theorem 3.17. *The functor $X \to \mathscr{E}\ell\ell_G^*(X)$ from finite G-CW complexes to graded rings is a stable G-equivariant cohomology theory.*

It is easy to show that the right-hand side of (3.14) is a stable G-equivariant cohomology theory. Theorem 3.17 follows immediately.

4. Completion theorems

4.1. Invariance properties of cohomology theories

Let G be a finite group, and let $\mathscr{F}$ be a family of subgroups which means that it is closed under passing to subgroups and conjugate subgroups. If X and Y are two G-CW complexes, then we shall say that a G-equivariant map $f : X \to Y$ is a *$\mathscr{F}$-equivalence* if the induced map of fixed point sets

$$f^H : X^H \to Y^H$$

is an ordinary homotopy equivalence for each subgroup $H \in \mathscr{F}$ [2, p. 7].

Example 4.1. Recall that a G-space $E\mathscr{F}$ is called a *universal space* for the family $\mathscr{F}$ if $E\mathscr{F}^H$ is contractible for $H \in \mathscr{F}$ and empty for $H \notin \mathscr{F}$; the construction of $E\mathscr{F}$ can be found in [33, Ch. 1, Section 6]. For any G-CW complex X the projection

$$p : E\mathscr{F} \times X \to X$$

is an $\mathscr{F}$-equivalence [2, p. 7].

Definition 4.2. Let $\mathscr{A}$ be an Abelian category. We shall say that a functor h from the category of G-CW complexes to $\mathscr{A}$ is *$\mathscr{F}$-invariant* if $h(f)$ is an isomorphism for every $\mathscr{F}$-equivalence $f : X \to Y$.

4.2. Pro-group valued cohomology theories

We shall describe briefly what we need about pro-groups; more details can be found in [2, Section 2; 5, Section 2]. Let $\mathscr{A}$ be a filtered category; for example an ordered set. Then an Abelian *pro-group $\boldsymbol{M}$ indexed by $\mathscr{A}$* is a contravariant functor from $\mathscr{A}$ to the category of Abelian groups. We shall write usually $\boldsymbol{M} = \{M_\alpha\}$, where the indices α are the objects of $\mathscr{A}$ and $M_\alpha = \boldsymbol{M}(\alpha)$. Let $\{M_\alpha\}$ and $\{N_\beta\}$ be two pro-groups. We define the set $\mathrm{ProHom}(\{M_\alpha\}, \{N_\beta\})$ of *pro-homomorphisms* from $\{M_\alpha\}$ to $\{N_\beta\}$ by

$$\mathrm{ProHom}(\{M_\alpha\}, \{N_\beta\}) = \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Hom}(M_\alpha, N_\beta),$$

where both limits are taken in the category of groups. The category **ProGr** whose objects are the pro-groups and whose morphisms are the pro-homomorphisms between pro-groups is an Abelian category [2, Section 2]. One can therefore define in the usual way pro-group valued cohomology theories.

4.3. Main results

We can associate to the functor $\mathscr{E}\ell\ell^*_G$ a pro-group valued G-cohomology theory ell^*_G defined on the category of G-CW complexes. If X is a G-CW complex, then

$$ell^*_G(X) = \{\mathscr{E}\ell\ell^*_G(X_\alpha)\},$$

where X_α runs over the finite G-sub-complexes of X. The morphisms

$$i^*_{\alpha\beta} : \mathscr{E}\ell\ell^*_G(X_\beta) \to \mathscr{E}\ell\ell^*_G(X_\alpha)$$

are induced by the inclusions $i_{\alpha\beta} : X_\alpha \to X_\beta$.

Let $\mathscr{F}$ be a family of subgroups of G. Then we can associate to $\mathscr{F}$ a second pro-group valued functor $X \to \mathscr{E}\ell\ell^*_G(X)\hat{}_{\mathscr{F}}$ defined on the category of G-CW complexes. This functor is defined by

$$\mathscr{E}\ell\ell^*_G(X)\hat{}_{\mathscr{F}} = \{\mathscr{E}\ell\ell^*_G(X_\alpha)/I_{\mathscr{F}}\mathscr{E}\ell\ell^*_G(X_\alpha)\},$$

where $I_{\mathscr{F}}$ runs over the finite products of the ideals I_H, defined in Section 2.2, for H an element of $\mathscr{F}$.

Remark 4.3. The generalizations of elliptic cohomology that we have defined can be also defined for every stable G-equivariant cohomology theory; see [1] for the case of equivariant K-theory.

Theorem 4.4. *The functor $X \to \mathscr{E}\ell\ell^*(X)\hat{}_{\mathscr{F}}$ is $\mathscr{F}$-invariant.*

Proof. Let us denote the reduced equivariant elliptic cohomology of a space X by $\widetilde{\mathscr{E}\ell\ell}^*_G(X)$. In order to prove the theorem it suffices, by [2, Lemma 2.2], to show that if X is a based G space such that X^H is contractible for all $H \in \mathscr{F}$, then

$$\widetilde{\mathscr{E}\ell\ell}^*_G(X)\hat{}_{\mathscr{F}} = \{\widetilde{\mathscr{E}\ell\ell}^*_G(X_\alpha)/I_{\mathscr{F}}\widetilde{\mathscr{E}\ell\ell}^*_G(X_\alpha)\}$$

is pro-zero, which means insomorphic to the zero object in the category **ProGr**; we refer to [2, p. 11] for an explicit description of pro-zero objects of **ProGr**.

If H is a subgroup of G, then we shall denote its conjugacy class by $[H]$. Recall that, using the $A(G)$-module structure of $\mathscr{E}\ell\ell^*_G$, we have obtained in Section 2.2 a decomposition $\mathscr{E}\ell\ell^*_G = \oplus_{[H]}\, e_H\mathscr{E}\ell\ell^*_G$. If K is a subgroup of G then

$$\mathrm{rest}^G_K(e_H\mathscr{E}\ell\ell^*_G) = \mathrm{rest}^G_K(e_H)\mathrm{rest}^G_K(\mathscr{E}\ell\ell^*_G),$$

where $\mathrm{rest}^G_K(e_H) \in A(K)$. From the description of the idempotents of the Burnside ring of [32, Ch. 1] it follows that if H is not conjugate to a subgroup of K, then $\mathrm{rest}^G_K(e_H) = 0$. Therefore

$$\bigoplus_{[H]\in S} e_H\mathscr{E}\ell\ell^*_G \subset I_K, \tag{4.1}$$

where the sum is over a set S of representatives of conjugacy classes of subgroups of G with the property that $[H] \in S$ if and only if $[H]$ is not conjugate to a subgroup of K.

Let f be the cardinal of $\mathscr{F}$ and let $H_1,\dots,H_f$ be a list of the subgroups of G in $\mathscr{F}$. We can form the pro-group

$$\mathbf{M} = \bigoplus_{[H]\subset\mathscr{F}} \mathbf{M}[H] = \left\{ \bigoplus_{[H]\subset\mathscr{F}} e_H \widetilde{\mathscr{E}\ell\ell}^*_G(X_\alpha) \right\}.$$

This system is indexed by the set of finite skeletons of X and, trivially, by the partially ordered set $\{(n_1,\dots,n_f) \mid n_i \geq 0\}$. By (4.1) there exists for each $(\alpha; n_1,\dots,n_f)$ with $n_i > 0$ an epimorphism

$$\oplus_{[H]\subset\mathscr{F}}\ e_H \widetilde{\mathscr{E}\ell\ell}^*_G(X_\alpha) \to \mathscr{E}\ell\ell^*_G(X_\alpha)/I_{H_1}^{n_1}\dots I_{H_f}^{n_f}\mathscr{E}\ell\ell^*_G(X_\alpha).$$

These epimorphisms induce an epimorphism $\mathbf{M} \to \mathscr{E}\ell\ell^*_G(X)\hat{}_{\mathscr{F}}$ in the category of pro-groups. Therefore it suffices to show that the system $\mathbf{M}$ is pro-zero. We shall show that each one of the systems $\mathbf{M}[H]$ is pro-zero. By Theorem 3.16 it suffices to consider the case $H = \langle g_1, g_2\rangle$ for some pair $(g_1,g_2) \in TG$. In this case

$$\mathbf{M}[H](\alpha, n_1,\dots,n_f) \sim [\widetilde{\mathscr{E}\ell\ell}^*(X_\alpha^{g_1,g_2}) \otimes_{\mathscr{E}\ell\ell^*} \mathscr{E}\ell\ell^*_{\langle g_1,g_2\rangle}]^{W\langle g_1,g_2\rangle}_{S(\langle g_1,g_2\rangle)}.$$

Milnor's exact sequence [25] for the space X^{g_1,g_2} gives us

$$\begin{aligned}
0 &\to \varprojlim{}^1 \mathbf{M}[H](\alpha, n_1,\dots,n_f) \\
&\to [\widetilde{\mathscr{E}\ell\ell}^*(X^{g_1,g_2}) \otimes_{\mathscr{E}\ell\ell^*} \mathscr{E}\ell\ell^*_{\langle g_1,g_2\rangle}]^{W\langle g_1,g_2\rangle}_{S(\langle g_1,g_2\rangle)} \\
&\to \varprojlim \mathbf{M}[H](\alpha, n_1,\dots,n_f) \to 0.
\end{aligned} \tag{4.2}$$

The first term in (4.2) is the first right derived functor of the inverse limit functor. By hypothesis X^{g_1,g_2} is contractible and therefore the middle term of the sequence (4.2) is zero. This implies that the inverse limit of the system $\mathbf{M}[H]$ is zero. Since the algebras $\mathbf{M}[H](\alpha, n_1,\dots,n_f)$ are finitely generated this implies that $\mathbf{M}[H]$ is pro-zero. □

Theorem 4.4 is a particular case of a "localization-completion" theorem which we shall describe now. If I is an ideal of $\mathscr{E}\ell\ell^*_G$ and S is a multiplicatively closed subset of $\mathscr{E}\ell\ell^*_G$, then we shall associate to the pair (I,S) the family of subgroups $\mathscr{P}$ defined by

$$\mathscr{P} = \bigcup \{\mathrm{Supp}(P) \mid P \cap S = \emptyset \text{ and } I \subset P\}. \tag{4.3}$$

Definition 4.5. If $\{M_\alpha\}$ is a pro-$\mathscr{E}\ell\ell^*_G$ module and S is a multiplicatively closed subset of $\mathscr{E}\ell\ell^*_G$ we define

$$S^{-1}\{M_\alpha\} = \{S^{-1}M_\alpha\}.$$

We can now state the localization-completion theorem.

Theorem 4.6. *The pro-group valued functor $X \to S^{-1}\mathscr{E}\ell\ell_G^*(X)\hat{}_{\mathscr{P}}$ defined on the category of G-CW complexes is $\mathscr{P}$-invariant.*

Proof. The proof of this theorem follows closely the proof of [1, Theorem 4.1] therefore we shall give only the general argument and provide details in the parts of the proof that are specific to elliptic cohomology. We refer the reader to [1, p. 5] for the rest of the details.

By general algebraic arguments [2, Lemma 2.3] it suffices to show that if X is a based G with the property that X^H is contractible for all $H \in \mathscr{P}$, then $S_P^{-1}\widetilde{\mathscr{E}\ell\ell}^*_G(X)\hat{}_P$ is pro-zero for each prime ideal $P \subset \mathscr{E}\ell\ell_G^*$ such that $P \cap S = \emptyset$ and $P \supset I$. The notation S_P^{-1} means "localization at P".

Let $H \in \operatorname{Supp} P$ and let $\mathscr{F}$ be the family of subgroups of G generated by H. Then we can embed X as a sub-complex of a G-CW complex Y which has the property that $Y^K = X^K$ for all K which contains a conjugate of H and Y^K is contractible for any other K [1]. By Theorem 4.4 $\mathscr{E}\ell\ell_G^*(Y)\hat{}_{\mathscr{F}}$ is pro-zero. It follows that $S_P^{-1}\mathscr{E}\ell\ell_G^*(Y)\hat{}_{\mathscr{F}}$ and, as by Corollary 2.28, P contains I_H, $S_P^{-1}\mathscr{E}\ell\ell_G^*(Y)\hat{}_P$ are both pro-zero. The classical localization results, see for example [32], imply that $S_P^{-1}\mathscr{E}\ell\ell_G^*(Y)\hat{}_P \to S_P^{-1}\mathscr{E}\ell\ell_G^*(X)\hat{}_P$ is a pro-isomorphism. This fact can also be proved from Theorem 3.16. Therefore $S_P^{-1}\mathscr{E}\ell\ell_G^*(X)\hat{}_P$ is pro-zero. □

If $\mathscr{F}$ is a family of subgroups of G and $E\mathscr{F}$ is the universal $\mathscr{F}$-free-G-space, then Theorem 4.6 has the following corollary.

Corollary 4.7. *If X is a finite G-CW complex, then the projection $E\mathscr{F} \times X \to X$ induces an isomorphism*

$$\mathscr{E}\ell\ell_G^*(X)\hat{}_{\mathscr{F}} \to \mathscr{E}\ell\ell_G^*(E\mathscr{F} \times X). \tag{4.4}$$

Proof. Let X be a finite G-CW complex. Then, by Theorem 3.2, it induces an isomorphism $\mathscr{E}\ell\ell^*(X)\hat{}_{\mathscr{F}} \to \mathscr{E}\ell\ell^*(E\mathscr{F} \times X)\hat{}_{\mathscr{F}}$. Using the description of equivariant elliptic cohomology given by the right-hand side of (3.14) it is easy to see that if Y is a finite G-CW complex such that all the isotropy groups are in $\mathscr{F}$, then $\mathscr{E}\ell\ell_G^*(Y)$ is annihilated by some power of $I_{\mathscr{F}}$ and hence $\mathscr{E}\ell\ell_G^*(Y)$ is $\mathscr{F}$-adically complete. As all the isotropy groups of the space $E\mathscr{F} \times X$ are in $\mathscr{F}$, the pro-groups $ell_G^*(E\mathscr{F} \times X)$ are $\mathscr{F}$-adically complete. On the other hand, due to the fact that X is a finite G-CW complex, the inverse system $\mathscr{E}\ell\ell_G^*(X)\hat{}_{\mathscr{F}}$ satisfies the Mittag–Leffler condition. This shows that the algebraic completion $\mathscr{E}\ell\ell_G^*(X)\hat{}_{\mathscr{F}}$ and topological completion $\mathscr{E}\ell\ell^*(E\mathscr{F} \times X)_F$ are isomorphic. □

In particular, taking as $\mathscr{F}$ the family formed by the trivial subgroup $\{e\}$ of G we obtain a generalization of the Atiyah–Segal completion theorem.

Corollary 4.8. *There exist, for X any finite G CW-complex X, an isomorphism*

$$\mathscr{E}\ell\ell_G^*(X)\hat{}_I \to \mathscr{E}\ell\ell_G^*(EG \times X), \tag{4.5}$$

where $I = \ker\{\mathrm{rest}^G_{\{e\}} : \mathscr{E}\ell\ell_G^* \to \mathscr{E}\ell\ell^*\}$.

If N is a normal subgroup of G and $\mathscr{J}$ is the family of subgroups H of G that satisfy $H \cap N = \{e\}$, then $E\mathscr{J} = E(N,G)$ [1].

Corollary 4.9. *If X is a finite G-CW complex where N acts freely, then the projection* $\pi : E(N,G) \times X \to X$ *induces an isomorphism*

$$\mathscr{E}\ell\ell_G^*(X)\hat{}_{\mathscr{J}} \to \mathscr{E}\ell\ell_G^*(E(N,G) \times X). \tag{4.6}$$

Combining these corollaries with a standard argument in equivariant topology, that implies that for a G space X where the normal subgroup N acts freely $\mathscr{E}\ell\ell_G^*(X) \simeq \mathscr{E}\ell\ell_{G/N}^*(X/N) \otimes [1/|G|]$ – we obtain a description of the elliptic cohomology of the spaces $EG \times_G X$ ($E(N,G) \times_N X$ respectively) for any finite G-CW complex (a finite G-CW complex with a free N action).

5. Relation with the work of Hopkins, Kuhn and Ravenel

5.1. Brief description of the results of Hopkins, Kuhn and Ravenel

Hopkins, Kuhn and Ravenel defined in [15] the notion of *generalized characters* of a finite group G; they used this notion to give, among other things, a description of a certain I-adic completion of the elliptic cohomology of the classifying space BG of G. We shall show in this section how some of these rings of "generalized characters", namely those that are associated with supersingular curves, are naturally related to our coefficient ring $\mathscr{E}\ell\ell_G^*$, and how the description of [15, Section 8] follows from our Corollary 4.9.

Let p be an odd prime, then we shall denote the p-adic integers by $\mathbb{Z}_p$, and we shall write $\bar{\mathbb{Q}}_p$ for the algebraic closure of the p-adic rationals. If G is a finite group, then we let $\mathrm{Hom}(\mathbb{Z}_p^n, G)$ be the set of group homomorphisms $\mathbb{Z}_p^n \to G$. The set $\mathrm{Hom}(\mathbb{Z}_p^n, G)$ admits an action of G given by

$$(g\alpha)(m_1,\dots,m_n) = g\alpha(m_1,\dots,m_n)g^{-1},$$

where $g \in G$, $\alpha \in \mathrm{Hom}(\mathbb{Z}_p^n, G)$, and $(m_1,\dots,m_n) \in \mathbb{Z}_p^n$.

Definition 5.1. The ring of *generalized characters* of level n is the ring

$$\mathrm{Cl}(\mathrm{Hom}(\mathbb{Z}_p^n, G),\ \bar{\mathbb{Q}}_p),$$

whose elements are the functions $f : \mathit{Hom}(\mathbb{Z}_p^n, G) \longrightarrow \bar{\mathbb{Q}}_p$ invariant under the action of G.

Remark 5.2. One can define refined characters using Galois theory; for the case $n = 1$ see Remark 2.3, where we used a description of $R(G) \otimes \mathbb{Q}$ as a ring of Galois equivariant class functions

$$\mathrm{Cl}(G, \mathbb{Q}(\zeta))^{G(\mathbb{Q}(\zeta)|\mathbb{Q})},$$

and [16, Proposition 1.5]; we shall describe here the case $n = 2$, and refer to [16] for the general case.

We shall briefly describe now [15, Corollaries 8.4 and 8.5]. Let $\mathcal{O}$ be the ring of integers in a finite extension $\mathbb{F}$ of the p-adic numbers $\mathbb{Q}_p$ with maximal ideal (π) and residue field $k = \mathcal{O}/(\pi)$. The basic data of the Hopkins–Kuhn–Ravenel construction is the choice of a ring homomorphism $\varphi : \mathcal{E}\ell\ell^* \to \mathcal{O}$ such that $\varphi(u_1) \subset (\pi)$, where u_1 is the coefficient of x^p in the p-series $[p]_E(x)$ associated to Euler's formal group law

$$E(x, y) = \frac{x\sqrt{1 - 2\delta y^2 + \varepsilon y^4} + y\sqrt{1 - 2\delta x^2 + \varepsilon x^4}}{1 - \varepsilon^2 x^2 y^2}.$$

Let E_φ be the Jacobi quartic of equation

$$y^2 = 1 - 2\varphi(\delta)x^2 + \varphi(\varepsilon)X^4 \tag{5.1}$$

defined over $\mathcal{O}$. The curve E_φ is naturally associated to the ring homomorphism φ. We shall denote the mod(π) reduction of E_φ by E_0. The mod p reduction of $\varphi(u_1)$ can be identified with the Hasse invariant of E_0 [15, 30].

If $\varphi(u_1) = 0 \bmod p$, then the Jacobi quartic E_φ has supersingular reduction at p. This implies that, for all $n \in \mathbb{N}$, E_0 has no non-trivial point of order p^n. The statement of Corollary 8.4 of [15] is that in this case

$$\mathcal{E}\ell\ell^*(BG)^\wedge \bigotimes_{\mathcal{E}\ell\ell^*} \bar{\mathbb{Q}}_p \simeq \mathrm{Cl}(\mathrm{Hom}(\mathbb{Z}_p^2, G), \ \bar{\mathbb{Q}}_p), \tag{5.2}$$

where for an $\mathcal{E}\ell\ell^*$ module M the expression $M^\wedge$ denotes the adic completion of M with respect to the ideal (p, u_1).

If $\varphi(u_1) \neq 0$, then the Jacobi quartic (5.1) has ordinary reduction. The statement of Corollary 8.5 of [15] is that in this case

$$\mathcal{E}\ell\ell^*(BG)^\wedge \bigotimes_{\mathcal{E}\ell\ell^*} \bar{\mathbb{Q}}_p \simeq \mathrm{Cl}(\mathrm{Hom}(\mathbb{Z}_p, G), \ \bar{\mathbb{Q}}_p). \tag{5.3}$$

This last corollary is really a statement about the theory $u_1^{-1}\mathcal{E}\ell\ell^*$ [15]. We shall therefore concentrate only in the supersingular case in which the best approximation to equivariant elliptic cohomology occurs.

5.2. The elliptic character ring

The ring $\mathrm{Cl}(G, \mathbb{Q}(\zeta))^{G(\mathbb{Q}(\zeta)|\mathbb{Q})}$ admits a natural generalization related to elliptic cohomology. Let $\mathbb{K}^*$ be the graded field of fractions of the ring $\mathcal{E}\ell\ell^*$ and let $\mathbb{K}^*(x)$ be the extension of $\mathbb{K}^*$ obtained by adjoining all the elements $S_{(a,b)}$, defined in (2.13), for $(a,b) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, where $n = |G|$ and $(a,b) \neq (0,0)$. It is not difficult to see that the extension $\mathbb{K}^*(x)$ is a Galois extension of $\mathbb{K}^*$ with Galois group $GL(2,\mathbb{Z}/n\mathbb{Z})$. The group $GL(2,\mathbb{Z}/n\mathbb{Z})$ acts on TG as in (2.9).

Definition 5.3. The *elliptic character ring* $\mathrm{Cl}(TG, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})}$ is the ring of functions

$$f : TG \to \mathbb{K}^*(x)$$

that are invariant under simultaneous conjugation and equivariant with respect to the actions of $GL(2,\mathbb{Z}/n\mathbb{Z})$ on TG and $\mathbb{K}^*(x)$.

There exists a natural morphism $ev : \mathscr{E}\ell\ell_G^* \to \mathrm{Cl}(TG, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})}$.

Proposition 5.4. *The homomorphism ev induces an isomorphism*

$$\mathscr{E}\ell\ell_G^* \otimes_{\mathscr{E}\ell\ell^*} \mathbb{K}^* \simeq \mathrm{Cl}(TG, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})}. \tag{5.4}$$

Proof. As $\mathbb{K}^*$ is a graded field and ev is a $\mathbb{K}^*$-linear monomorphism it suffices to check that both sides of (5.4) have the same $\mathbb{K}^*$ rank. The rank of the left-hand side have been computed in [12] where we showed that it is equal to

$$\chi_{\mathscr{E}\ell\ell} = \frac{1}{|G|} \#\{(g_1,g_2,g_3) \in G \times G \times G \mid g_i g_j = g_j g_i;\ i,j = 1,2,3\}. \tag{5.5}$$

The rank of the right-hand side can be computed as follows. Let

$$TG' = \{(g_1,g_2)_0, \ldots, (g_1,g_2)_n\}$$

be a complete set of representatives of the orbits of the action of $GL(2,\mathbb{Z}/n\mathbb{Z}) \times G$ on TG. We shall write $S_i \subset GL(2,\mathbb{Z}/n\mathbb{Z}) \times G$ for the isotropy group of $(g_1,g_2)_i$, Γ_i for the isotropy subgroup of $(g_1,g_2)_i$ in $GL(2,\mathbb{Z}/n\mathbb{Z})$, and $\Gamma_i^1 = p(S_i)$, where $p : GL(2,\mathbb{Z}/n\mathbb{Z}) \times G \to GL(2,\mathbb{Z}/n\mathbb{Z})$ is the projection.

It is easy to see that

$$\mathrm{rank}_{\mathbb{K}^*} \mathrm{Cl}(TG, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})} = \sum_i \mathrm{rank}_{\mathbb{K}^*} \mathbb{K}^*(x)^{\Gamma_i^1}.$$

Using Galois theory we see that $\mathrm{rank}_{\mathbb{K}^*} \mathbb{K}^*(x)^{\Gamma_i^1} = [GL(2,\mathbb{Z}/n\mathbb{Z}), \Gamma_i^1]$.

We have an exact sequence

$$0 \to C_{gg}(G) \to S_i \to \Gamma_i^1 \to 0,$$

where $C_{gg}(G)$ is the centralizer of g_1 and g_2 in G. Using this exact sequence one can see that $|S_i| = |\Gamma_i^1||C_{gg}(G)|$, and therefore the cardinal of the orbit of $(g_1,g_2)_i$ is equal to

$$\frac{|GL(2,\mathbb{Z}/n\mathbb{Z})||G|}{|S_i|} = \frac{|GL(2,\mathbb{Z}/n\mathbb{Z})||G|}{|\Gamma_i^1||C_{gg}(G)|}.$$

Then we have

$$\begin{aligned}\operatorname{rank}_{\mathbb{K}^*} \mathrm{Cl}(TG, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})} &= \sum_i \frac{|GL(2,\mathbb{Z}/n\mathbb{Z})|}{|\Gamma_i^1|} \\ &= \sum_{(g_1,g_2)\in TG} \frac{|GL(2,\mathbb{Z}/n\mathbb{Z})|}{|\Gamma_i^1|} \frac{|\Gamma_i^1||C_{gg}(G)|}{|GL(2,\mathbb{Z}/n\mathbb{Z})||G|} \\ &= \frac{1}{|G|} \sum_{(g_1,g_2)\in TG} |C_{gg}(G)| = \chi_{\mathscr{E}\ell\ell}. \qquad \square\end{aligned}$$

5.3. Elliptic curves over local fields

Let us recall some of the relevant aspects of the arithmetic of elliptic curves over local fields. These results are all well known and can be found, for example, in [30].

Let K be a local field that is complete for a discrete valuation v; we shall denote the ring of integers of K by A, the maximal ideal by (π), and the residue field by k. Let E be an elliptic curve defined over A that has good reduction $E_\pi \bmod \pi$; we shall write E_0 for the group of torsion points of E whose reduction mod p is the identity element of E_π. If F_E is the formal group law associated to the elliptic curve E, then F_E induces a group structure on π, which we shall denote by π_E, and the torsion part of this group is canonically isomorphic to the group E_0 [30]. This result is also valid for a Jacobi quartic, provided that $p = \operatorname{char} k \neq 2$ and that we restrict ourselves to torsion points of odd order.

5.4. $\{\mathscr{E}\ell\ell_G^*\}\hat{}$ and the generalized characters of Hopkins–Kuhn–Ravenel

Let $TG_p = \operatorname{Hom}(\mathbb{Z}_p \times \mathbb{Z}_p, G)$. Then TG_p can be identified with the elements $(g_1, g_2) \in TG$ such that the orders of g_1 and g_2 are powers of p. With this identification TG_p is a $GL(2,\mathbb{Z}/n\mathbb{Z}) \times G$-invariant subset of TG and therefore the inclusion $TG_p \subset TG$ induces an homomorphism

$$\mathrm{Cl}(TG, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})} \xrightarrow{i_p'} \mathrm{Cl}(TG_p, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})}. \tag{5.6}$$

Let $\mathbb{K}^*(x_p)$ be the subfield of $\mathbb{K}^*(x)$ obtained by adjoining the elements $S_{(a,b)}$ with a and b of order a power of p. Then, due to the $GL(2,\mathbb{Z}/n\mathbb{Z})$-equivariance of the elements of the elliptic character ring, the map i_p' admits a factorization

$$\begin{aligned}&\mathrm{Cl}(TG, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})} \\ &\quad \xrightarrow{i_p} \mathrm{Cl}(TG_p, \mathbb{K}^*(x_p))^{GL(2,\mathbb{Z}/n\mathbb{Z})} \to \mathrm{Cl}(TG_p, \mathbb{K}^*(x))^{GL(2,\mathbb{Z}/n\mathbb{Z})}.\end{aligned} \tag{5.7}$$

Let $\gamma : \mathrm{Cl}(TG_p, \mathbb{K}^*(x_p))^{GL(2,\mathbb{Z}/n\mathbb{Z})} \to \mathrm{Cl}(TG_p, \mathbb{K}^*(x_p))^{GL(2,\mathbb{Z}/p\mathbb{Z})}$ be the natural homomorphism and let $\mathbb{K}_p^*$ be the (p, u_1)-adic completion of $\mathbb{K}^*$. Then the evaluation homomorphism composed with the homomorphism γi_p and completion induces an homomorphism

$$\{\mathscr{E}\ell\ell_G^*\}\hat{}\otimes \mathbb{K}_p^* \xrightarrow{ev_p} \mathrm{Cl}(TG_p, \mathbb{K}_p^*(x))^{GL(2,\mathbb{Z}/p\mathbb{Z})}. \tag{5.8}$$

The homomorphism φ has a unique extension to an homomorphism $\varphi : \mathbb{K}_p^* \to \mathbb{F}$ and if $\mathbb{F}(x)$ is the extension of $\mathbb{F}$ that we obtain if we adjoin the elements $S^{\varphi}_{(a,b)}$, $(a,b) \in \mathbb{Z}/p^l\mathbb{Z} \times \mathbb{Z}/p^l\mathbb{Z} - \{(0,0)\}$ determined by the quartic (5.1) considered as a curve over $\mathbb{F}$, where p^l is the order of a p Sylow subgroup of G, then we have a (non-canonical) extension $\varphi : \mathbb{K}_p^*(x) \to \mathbb{F}(x)$. Using this extension we obtain from (5.4) an homomorphism

$$\{\mathscr{E}\ell\ell_G^*\}\hat{}\otimes \mathbb{K}_p^* \xrightarrow{ev_p} \mathrm{Cl}(TG_p, \mathbb{F}_p(x))^{GL(2,\mathbb{Z}/p\mathbb{Z})}.$$

5.5. Supersingular reduction

Let us suppose now that (5.1) has supersingular reduction. In this case there exists an isomorphism $\imath$ between the group of points of order p^j of π_E and the group of p^j-torsion points of E. Then if $\mathbb{F}(x')$ is the extension of $\mathbb{Q}_p$ obtained by adjoining the points of order p^l of π_E $\imath$ induces an isomorphism $\mathbb{F}(x) \simeq \mathbb{F}(x')$. Taking the composition of this isomorphism with the inclusion $\mathbb{F}(x') \to \overline{\mathbb{Q}}_p$ we obtain an homomorphism

$$\mathrm{Cl}(TG_p, \mathbb{F}(x))^{GL(2,\mathbb{Z}/p^l\mathbb{Z})} \to \mathrm{Cl}(TG_p, \overline{\mathbb{Q}}_p).$$

In this way we obtained an homomorphism

$$\mathscr{E}\ell\ell_G^{*}\hat{}\otimes \mathbb{K}_p^* \to \mathrm{Cl}(TG_p, \overline{\mathbb{Q}}_p). \tag{5.9}$$

The evaluation map ev sends I_G into I_i, where I is the kernel of i'_p (see 5.6). Using the characteristic function of the set TG_p it is not difficult to see that $I_i^n = I_i$. From this it follows that the homomorphism (5.9) induces an homomorphism

$$\mathscr{E}\ell\ell^*(BG)\hat{}\otimes \overline{\mathbb{Q}}_p \to \mathrm{Cl}(TG_p, \overline{\mathbb{Q}}_p). \tag{5.10}$$

Then [15, Corollary 8.4] is the statement that this homomorphism is an isomorphism.

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